The Inherent Position Errors in Double-Theodolite Pibal Measurements

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ABSTRACT

A simple analysis of the position error inherent in double-theodolite pibal systems is presented. The quality of data collected by double theodolites is very sensitive to the geometric design of the system, and care must be taken in the interpretation of results.

1. Introduction

It is all too common for those who use real data in scientific calculations to fail to consider the inherent limitations of the data. This practice arises, in part, through the ready availability of digital computers. However, even some of the problems encountered by Richardson (1922) in hand calculations resulted from the finite reliability of the input data. Thus, this practice has a long history and most, if not all, of us commit this error at times in our haste to see results.

Neoprene balloons are the primary vehicle for nonsurface meteorological wind data collection. The balloon ascent rate impacts not only wind data derived from pilot balloons, but also in-thunderstorm vertical velocity estimates (e.g., Davies-Jones and Henderson, 1975). Recently a controversy over the ascent rate of such balloons has arisen in the literature (Boatman, 1974, 1975; Nelson, 1975; Mansell, 1976; Murray and Auer, 1976).

These papers are based upon double-theodolite measurements of balloon position. From these measurements, estimates of the three-dimensional balloon velocity have been obtained. The controversy centers on the apparent dependence of the balloon-derived vertical velocity on the atmosphere's thermal stratification. We will not attempt to resolve this controversy here, but rather point out the limitations of double-theodolite systems as dictated by geometric considerations.

2. A brief history of double-theodolite pilot balloon measurements

Before proceeding to a simple error analysis for double-theodolite measurements, it is instructive to review some of the substantial number of papers on the subject of errors in pilot balloon measurement of winds.

Among the first estimates of the vector error in winds derived from double-theodolite measurements are those of Arnold (1948). When tracking the same balloon with "two distinct and separate double theodolite observations," the average wind speeds obtained by the two systems differed by 0.5 m s^{-1} and the computed height of the balloon varied by an average of 10 m. Since the baseline of his theodolite system is not known, a geometric interpretation of these results cannot be made. In discussing double-theodolite systems, the War Department (1945) admonished observers to use a baseline length of about 1 mi, oriented as nearly perpendicular to the mean wind as possible. Presumably this is to minimize errors in the calculations. Middleton and Spilhaus (1953) repeat this admonition in a detailed discussion of the problem's geometry. Also, Hansen and Taft (1959) point out the cumulative nature of position errors, suggesting that the errors are an increasing function of height.

These papers are based on the so-called "three-angle" method. Each of the two theodolites gives readings of an elevation and an azimuth angle. However, the balloon's position in space can be specified by using only three angles. Theoretically, the fourth angle is functionally dependent upon the other three. If errors are present in the angular readings, the three-angle method can give two different balloon elevations, depending on which angles are used in the computations. Also, when the balloon crosses the baseline, the balloon position becomes indeterminate, and an independent two-angle formula must be used.

In an effort to overcome these problems, Thyer (1962) devised a "four-angle" method of evaluating double-theodolite data. This method is based on the interdependence of the four angular measurements. The direction cosines of the rays from each theodolite to the balloon are computed. The length and orientation of the line perpendicular to both rays are then calculated and the "most probable" position of the balloon on this "short line" is determined. Thyer shows that the length of the short line depends on the length of the two rays and the accuracy of the angular measurements.

3. Geometric considerations

Consider the double-theodolite configuration shown in Fig. 1. Points T_1 and T_2 are the theodolite locations (assumed, for simplicity only, to have the same elevation), separated by a baseline of length b. The rays from theodolites T_1 and T_2 to the balloon (located

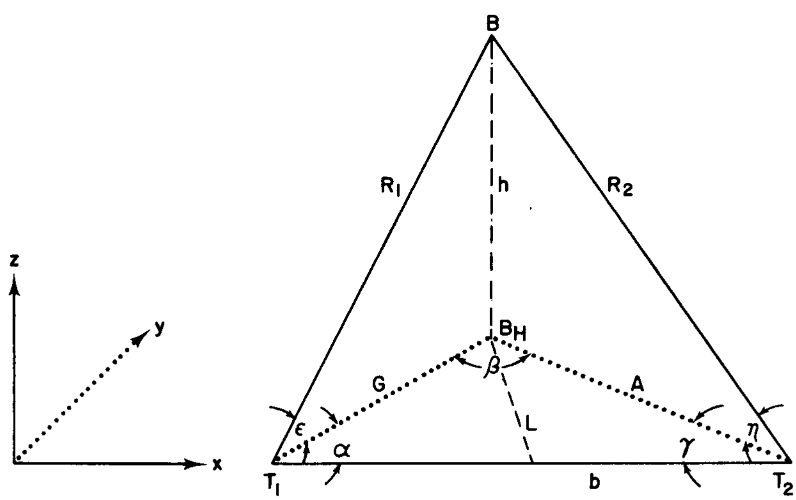

FIG. 1. Schematic of double-theodolite geometry. Notation is explained in text.

at B) have lengths R_1 and R_2, respectively. The azimuth and elevation angles measured by theodolite T_1 are α and ϵ, while those measured by T_2 are γ and η.

The point designated B_H is the projection of the balloon position onto the horizontal plane. The horizontal displacement of the balloon from its release point (assumed, again only for clarity, to be at the middle of the baseline) is L and the height of the balloon is h. When the three-angle method is used, the coordinates of B_H are computed first. The horizontal triangle $T_1T_2B_H$, has sides A, b, G and from the law of sines it is known that

$$\frac{\sin\alpha}{A}=\frac{\sin\gamma}{G}=\frac{\sin\beta}{b}=\frac{\sin(\alpha+\gamma)}{b}. \quad (1)$$

Thus, the theodolite azimuth angles determine the length of side G as

$$G(\alpha,\gamma)=\frac{b\sin\gamma}{\sin(\alpha+\gamma)}. \quad (2)$$

Analytic geometry dictates that the (x,y) coordinates of B_H from an origin located at T_1 are $(G\cos\alpha, G\sin\alpha)$. If errors $\delta\alpha$ and $\delta\gamma$ exist in the angular measurements, the coordinates of the projection point will be in error by

$$\delta x=x\{[-\tan\alpha-\cot(\alpha+\gamma)]\delta\alpha +[\cot\gamma-\cot(\alpha+\gamma)]\delta\gamma\}, \quad (3)$$

$$\delta y=y\{[\cot\alpha-\cot(\alpha+\gamma)]\delta\alpha +[\cot\gamma-\cot(\alpha+\gamma)]\delta\gamma\}. \quad (4)$$

These errors can be caused either by experimental problems or even by the limited precision of the theodolite system. Further interpretation is greatly simplified if it is assumed that $\delta\alpha=\delta\gamma$. While, in general, these azimuth errors are independent, if they are considered to be caused only by precision problems, this is not an unreasonable restriction. When this is the case, the positioning of the horizontal projection of the balloon will be in error by

$$\delta L=(\delta x^2+\delta y^2)^{\frac{1}{2}} =\frac{b\sin\gamma}{\sin(\alpha+\gamma)}\{1+[\cot\gamma-2\cot(\alpha+\gamma)]^2\}^{\frac{1}{2}}\delta\alpha. \quad (5)$$

To illustrate this effect, Eq. (5) has been solved numerically for $\delta L/b=10\%$. The results for $\delta\alpha=0.1°$ and $\delta\alpha=1°$ are shown in Fig. 2. Horizontal projection points lying outside the infinity sign have a displacement error greater than 10% of the baseline length. That means for a 1 km baseline, with azimuth readings accurate to the nearest tenth of a degree, it is impossible to determine the horizontally projected position of a balloon within 100 m if the balloon is over 5.4 km from the baseline. For other positions of the balloon, displacement errors reach the 10% baseline level more quickly.

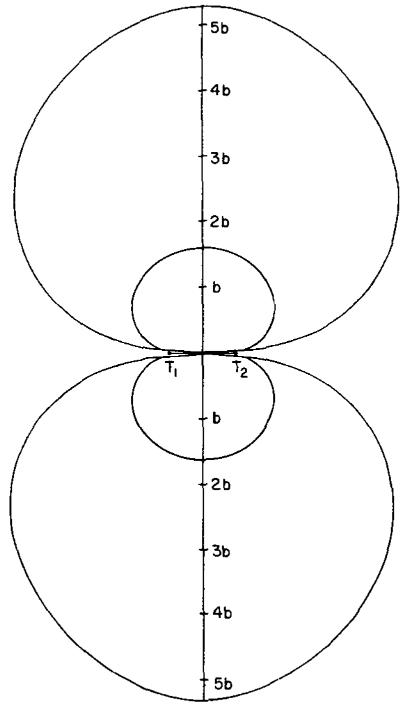

FIG. 2. Regions where displacement error is less than 10% of the baseline length. T_1 and T_2 denote positions of the theodolites. Outer figure is for $\delta\alpha=\delta\gamma=0.1°$; inner figure is for $\delta\alpha=\delta\gamma=1.0°$.

If the positioning errors are random, a crude estimate of the accuracy of the speed computed from the measurements can be obtained. If two measurements are made with sampling error δ_1 and δ_2, the difference between the measurements has a sampling error of $\Delta=(\delta_1^2+\delta_2^2)^{\frac{1}{2}}$. For a balloon released from the midpoint of a 1 km baseline under the influence of a constant 10 m s^{-1} wind and position measured with theodolites having an accuracy of 0.1°, the position uncertainty after 3 min is 12 m and after 4 min 21 m. Thus the wind speed calculated at $3\frac{1}{2}$ min has an expected inaccuracy up to 0.4 m s^{-1}. However, by $8\frac{1}{2}$ min, the position uncertainty has risen to 92 m, and by $9\frac{1}{2}$ min it is 114 m, yielding a speed uncertainty after 9 min of 2.4 m s^{-1}.

As the accuracy of the measurements degenerates, the domain of acceptable readings rapidly diminishes. Readings only 1° in error force the 10% baseline accuracy limit to move within 1.7 baselines for the best case.

As long as α does not equal zero, the "three-angle" method allows the height of the balloon to be calculated by either

$$h=G\tan\epsilon=\frac{b\sin\gamma}{\sin(\alpha+\gamma)}\tan\epsilon \quad (6)$$

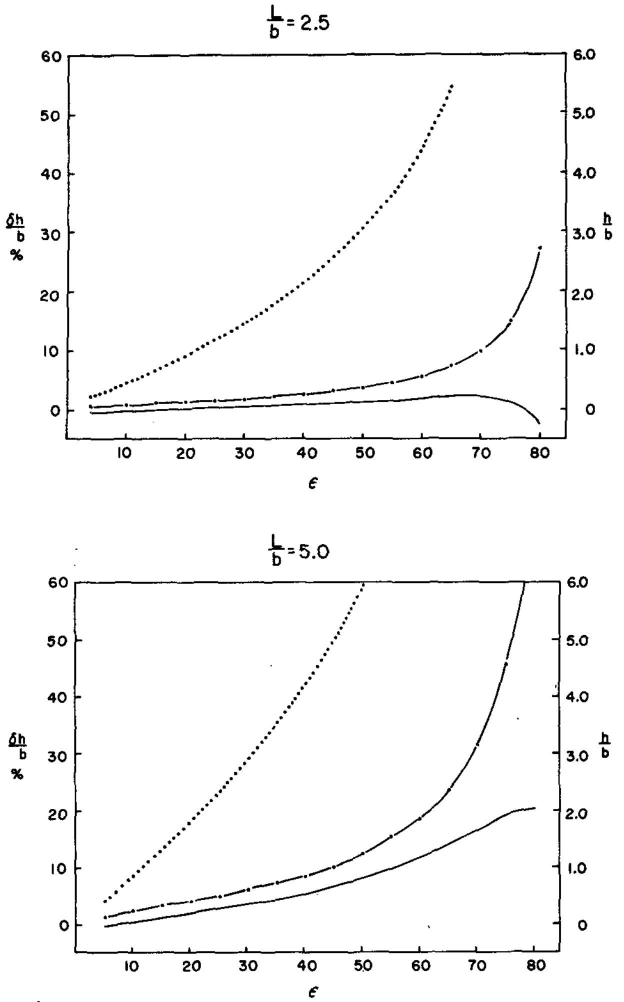

FIG. 3. Height and height error relative to baseline length as a function of elevation angle for the "best" case when the balloon is at (a) 2.5 baseline lengths and (b) 5.0 baseline lengths. Dotted curve is for height and is scaled with right-hand ordinate. Dash-dotted curve is for a positive correlation of the azimuth and elevation angle measurement error ($\delta\alpha = 0.1° = \delta\epsilon$) and solid curve for a negative correlation ($\delta\alpha = 0.1° = -\delta\epsilon$); both are scaled with left-hand ordinate.

or

$$h = A \tan\eta = \frac{b \sin\alpha}{\sin(\alpha+\gamma)} \tan\eta. \quad (7)$$

In full generality, the height error (δh) obtained from (6) is

$$\delta h = h\{[\cot\gamma - \cot(\alpha+\gamma)]\delta\gamma - \cot(\alpha+\gamma)\delta\alpha + \sec\epsilon \csc\epsilon\delta\epsilon\}. \quad (8)$$

The result obtained from (7) is similar except the α's and γ's are interchanged and η is used instead of ϵ. Thus, significant height errors can arise from inexact readings of either the azimuth or the elevation angles. Since all of the trigonometric functions in (8) are unbounded, the height errors in actual conditions can attain sufficient magnitude to make the computation of vertical velocity nearly meaningless.

A qualitative indication of the height error can be most easily obtained by once again considering the best case, i.e., the balloon is positioned over the perpendicular bisector of the baseline. In this situation

$$2\alpha + \beta = 180°, \quad (9)$$

$$G = L \csc\alpha = \tfrac{1}{2}b \sec\alpha. \quad (10)$$

If the two azimuth errors are considered to be equal, the error in length G, the horizontal projection of ray R_1, can be expressed as

$$\delta G = \tfrac{1}{2}b \sec\alpha \tan\alpha\, \delta\alpha. \quad (11)$$

Differentiation of (6) then gives the height error as

$$\delta h = \tfrac{1}{2}b \sec\alpha(\tan\epsilon \tan\alpha\, \delta\alpha + \sec^2\epsilon\, \delta\epsilon). \quad (12)$$

For purposes of illustration, plots of $\delta h/b$ as calculated via (12) are shown in Fig. 3 for two horizontal projection points ($L/b = 2.5$ and $L/b = 5.0$). The further simplification is made that $\delta\alpha = 0.1° = \delta\epsilon$, in which case the azimuth and elevation angle measurements are positively correlated and of the same precision. These restrictive assumptions are not meant to suggest that they are applicable for a particular set of measurements, but are intended to allow us to determine a representative value for $\delta h/b$. Results support the intuitive concept that the height error for a given horizontal distance along the bisector increases with elevation angle. Conversely, for a given elevation angle, the error increases with distance from the baseline (i.e., with increasing α).

Also shown are the height errors for a negative correlation of azimuth and elevation angle measurement error ($\delta\alpha = 0.1° = -\delta\epsilon$). As may be seen, the height error in this case is somewhat smaller in magnitude, becoming negative at large elevation angles. The correlation between $\delta\alpha$ and $\delta\epsilon$ is not known *a priori* to be either positive or negative for a given set of measurements; nor is it obvious that $\delta\alpha$ and $\delta\epsilon$ need to have the same magnitude. The intrinsic complexity of the geometry precludes an easy interpretation of this result, but one might suspect that the difference between positive and negative correlation curves is qualitatively related to the expected *range* of possible height errors under the given angular precision of the measurements. This range is compatible with that given by Arnold (1948).

An analysis of Thyer's four-angle method is considerably more complicated. However, for the ideal case considered above, Thyer's technique collapses to the three-angle method. Therefore, the height errors due to either computing technique will be similar for the best case. It is worth noting that while the concept behind Thyer's formula for the maximum length of the short line is correct, it does not give an adequate estimate of the error. This is because the measurement of the lengths of rays R_1 and R_2 has an error which is dependent on the accuracy of the angular measurements. The maximum length of the short line is

$$\delta = 0.05\ (R_1 + \delta R_1 + R_2 + \delta R_2)\ 57.2958. \tag{13}$$

For the ideal case, it can easily be seen that

$$R_2 = R_1 = \tfrac{1}{2} b \sec\epsilon \sec\alpha = G \sec\epsilon \equiv R,$$

and the error in ray length, as a result of angle measurement error, is then

$$\delta R = \tfrac{1}{2} b [\sec\epsilon \sec\alpha \tan\alpha\ \delta\alpha + \sec\alpha \sec\epsilon \tan\epsilon\ \delta\epsilon] = R[\tan\alpha\ \delta\alpha + \tan\epsilon\ \delta\epsilon].$$

The importance of using (13) instead of Thyer's formula for error estimation can be seen by examining Nelson's (1973) results. His computed short-line lengths exceeded the maximum error calculated via Thyer's formula by an order of magnitude. This difference can be accounted for by including the δR's, as in (13).

4. Implications

While the examples considered in depth are extremely restrictive, a basic problem in determining balloon position via double-theodolite measurements has been illustrated. Simply having dual-theodolite systems is not a panacea for obtaining high-quality data. Since the accuracy of the data is intimately and nonlinearly related to both baseline length and the range of the balloon, the purpose of the experiment must be known before the measurement system is deployed. That is, the absolute error (δh or δL) is proportional to b, so that for mathematically similar triangles (the relative errors are equal), the *absolute* error is smaller when b is smaller. However, if the balloon has traveled a significant distance from the baseline, the relative error is substantially greater with a short baseline. If results are to emphasize the early parts of the balloon run, when the balloon is close to the baseline, it might be advantageous to use a short baseline. On the other hand, if a short baseline results in the balloon being several baseline lengths away at the time of interest, it is likely that a longer baseline is called for.

Even though we have specifically addressed the dual-theodolite problem, the geometry is applicable to any data collection system which is based upon triangulation. Stereo-photogrammetry (e.g., Kassander and Sims, 1957) and dual-Doppler radar systems (e.g., Ray and Wagner, 1976) are two other such systems commonly used in meteorology. What we wish to stress is that caution must be taken when collecting and interpreting data.

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